Indian Statistical Institute, Bangalore

B. Math. First Year First Semester

Analysis -I

Mid-term Examination | Date: 11-09-2017

Maximum marks: 100 | Time: 3 hours

Instructor: B V Rajarama Bhat

(1) Show that the set of polynomials with integer coefficients:

$$\mathcal{P} = \{p : p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,\ n \in \mathbb{N} \bigcup \{0\}, a_j \in \mathbb{Z}, 0 \leq j \leq n, a_n \neq 0\}$$

is countable. [15]

(2) Prove that the set of natural numbers is not bounded above. [15]

(3) Let $I_n = [a_n, b_n]$ be a nested family of closed intervals, that is, $a_n, b_n \in \mathbb{R}, a_n < b_n$, with $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for every n. Assume that $\lim_{n\to\infty}(b_n - a_n) = 0$. Suppose $\{x_n\}_{n\geq 1}$ is a sequence of real numbers, where $x_n \in I_n$ for every n. Show that $\{x_n\}_{n\geq 1}$ is convergent. [15]

(4) Find lim sup and lim inf of following sequences of real numbers (Do provide proofs of your claims):

(i) $\{a_n\}_{n\geq 1}$ where $a_n = \frac{1}{n+1} - \frac{1}{n}$ for $n \in \mathbb{N}$;

(ii) $\{b_n\}_{n\geq 1}$ where

$$b_n = \begin{cases} 3 & \text{if } n = 3k-2, k \in \mathbb{N} \\ 5 + \frac{10}{n} & \text{if } n = 3k-1, k \in \mathbb{N} \\ 6 & \text{if } n = 3k, k \in \mathbb{N} \end{cases}$$

[5+10]

(5) Show that a sequence $\{a_n\}_{n\geq 1}$ of real numbers is convergent if and only if it is Cauchy. [15]

(6) Find the set of cluster points of following subsets of $\mathbb{R}$:

(i) $A = \{2 + \frac{(-1)^n}{7n} : n \in \mathbb{N}\}$;

(ii) $B = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$.

[5+10]

(7) Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$|h(x) - h(y)| \leq 10|x - y|,\ \forall x, y \in \mathbb{R}.$$

Show that h is continuous. [15]